A Taste of (formal) Sound Reasoning

A Tutorial

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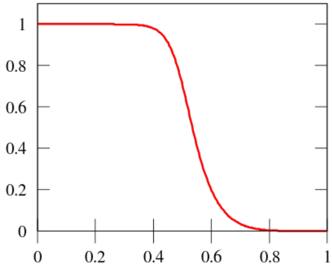
https://github.com/ejgallego/mini-faust-coq

Let's start with a simple IIR filter:

$$\text{smooth}_n = (1 - c) \cdot x_n + c \cdot \text{smooth}_{n-1}$$



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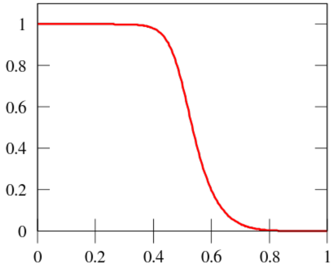

What can we know about it?

$$\text{smooth}_n = (1 - c) \cdot x_n + c \cdot \text{smooth}_{n-1}$$

Natural questions are:

- Frequency response;
- Stability;
- Linearity/Time Invariance.

Answers given by standard DSP theory.

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We dive into the realm of PL theory!

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Paradigm shift!

Certainty

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How do we know something is "absolutely" true?

Many possible answers

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In the Programming Languages field, we want computers to check knowledge for us!

How does it work?

Welcome to Evidence!

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We will build a particular kind of evidence for a property of our filter, then use the computer to validate it.

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Bob	Hi Alice, my dog is feeling weird!
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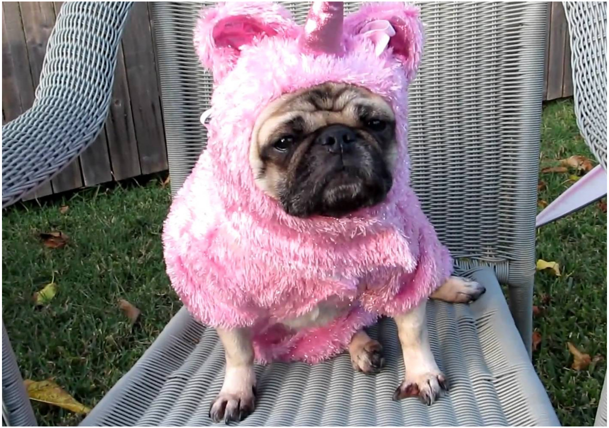

Logical Evidence: Proofs

We want to agree on a convention to produce and check evidence.

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Proposition Examples

"Every even number is not prime."
"Every complex polynomial has a root."
"Every finite impulse filter is stable."

Checking Validity: Inference

To check when a proposition holds, we need rules.

Rule Examples

"If A and B hold, B holds."

"If P holds for 0, and assuming P holds for n we can prove that P holds for n+1, then P holds for all n."

The Theory of Forms

Truth

Truth lives in the idealistic, infinite universe.

$$\Gamma \models \varphi \qquad \text{if } \Gamma \text{ is true, then } \varphi \text{ is true}$$

Proof

Reasoning lives in the concrete, syntactic universe.

$$\Gamma \vdash \varphi \qquad \varphi \text{ can be proved from } \Gamma$$

using a valid application of the rules.

Linking the Worlds

$\Gamma \models \varphi$	if Γ is true, then φ is true
$\Gamma \vdash \varphi$	φ can be proved from Γ

Main Properties

- Soundness: $\Gamma \vdash \varphi$ implies $\Gamma \models \varphi$.

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We are liberated from the complexity of the ideal, infinite world; we can now use mechanical, finitary rules to reason about it!

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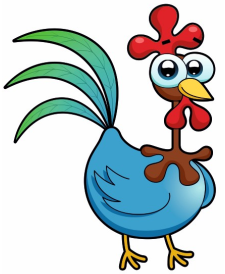
A crucial, fundamental idea:

Programs are Proofs!
Types are Propositions!

Computational Evidence

Welcome to Coq!

```
aptitude install coq
```



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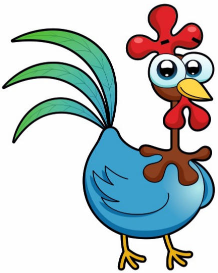

In Coq, proofs are precisely the well-typed functional programs. Type-checking validates our logical deductions!

BHK-Interpretation

Computational interpretation of logic

Type	**Proof / Program**
$P \wedge Q$	Record with proofs of P and Q.
$P \rightarrow Q$	Program that takes a proof of P, then produces a proof of Q.
$\forall(x : P), Q(x)$	Program that takes p, a proof of P, then produces a proof of $Q(p)$
$\exists(x : P), Q(x)$	Pair (p, q) of p, a proof of P, and q, a proof of $Q(p)$.
$P \vee Q$	????

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Use Coq to reason about audio programs; in particular, we’ll use a toy version of Faust!

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[Seriously, we'd love to hear about 4!]

Back to the Filter

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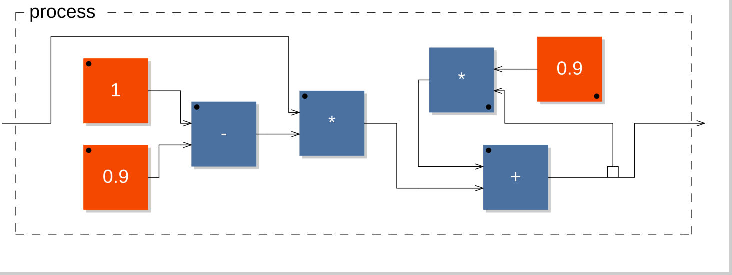
Using Faust:

```
smooth(c) = *(1-c) : + ~ *(c)
```



[For c = 0.9]

Let's do it!

```
smooth(c) = *(1-c) : + ~ *(c)
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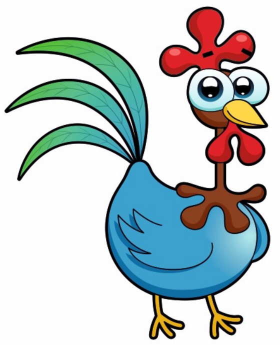

Semantics

We can “write” Faust programs inside Coq. Now we want to run them.

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Output of Smooth

T:	1	2	3	4	5	6	7	8
I:	1.00	1.05	1.10	1.15	1.20	1.25	1.20	1.25
O:	0.10	0.19	0.28	0.37	0.45	0.53	0.61	0.68

What is Sound? Choices…

We need to choose how to represent sound in Coq? In the formal world, we *pay* for every detail.

- Conceptual representations? ($\mathbb{R} \to \mathbb{R}$).
- Infinite representations? ($\mathbb{N} \to \mathbb{R}$)
- Finite representation? (seq $\mathbb{R}$)

We'll use the last one.

Let's do it!

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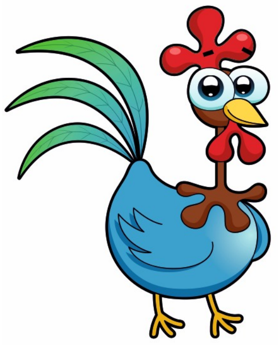

When is Smooth Stable?

We are in good shape; now, when is smooth stable?

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Smooth is stable when $c \in (0, 1)$. Formally:

$$\forall i \in [a, b],\ c \in [0, 1] \rightarrow smooth(c)\ i \in [a, b]$$

Proving Stability

We can do the proof directly in Coq; it is not difficult but cumbersome in general.

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But we can use better, higher-level reasoning principles: Use *program logics* and target global properties over all samples.

Sampled Logic

Definition

A sample-level property φ holds for a signal *s* if it holds for all samples: $\forall n.\varphi(s[n])$.

Boundedness is a sample-level property!

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Definition

Assume a program *f*, then we write $\{\varphi\}\ f\ \{\psi\}$ for "for all inputs satisfying φ, the output of *f* satisfies ψ."

Stability for smooth is written:

$$\{x \in [a, b]\} \text{ smooth } \{x \in [a, b]\}$$

Sampled Logic

$$\frac{\forall i_1, i_2, (\varphi_1(i_1) \wedge \varphi_1(i_2)) \implies \psi(i_1 + i_2)}{\{\varphi_1, \varphi_2\} + \{\psi\}}\,\textit{Prim}$$

$$\frac{\{\varphi\}\ f\ \{\theta\} \qquad \{\theta\}\ g\ \{\psi\}}{\{\varphi\}\ f : g\ \{\psi\}}\,\textit{Comp}$$

$$\frac{\models \psi(x_0) \qquad \{\theta, \varphi\}\ f\ \{\psi\} \qquad \{\psi\}\ g\ \{\theta\}}{\{\varphi\}\ f \sim g\ \{\psi\}}\,\textit{Feed}$$

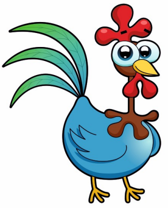
Stability Proof

$$
\dfrac{\dfrac{\Box}{\{I_{ab}\} \ast(1-c) \ \{I_{ab\bar{c}}\}} \qquad \dfrac{\dfrac{\Box}{\{I_{abc}, I_{ab\bar{c}}\} + \{I_{ab}\}} \qquad \dfrac{\Box}{\{I_{ab}\} \ast(c) \ \{I_{abc}\}}}{\{I_{ab\bar{c}}\} + \sim \ast(c) \ \{I_{ab}\}}}{\{i \in [a,b]\} \ast(1-c) : + \sim \ast(c) \ \{o \in [a,b]\}}
$$

with:

$$
\begin{aligned}
I_{ab}(x) &\equiv x \in [a, b] \\
I_{abc}(x) &\equiv x \in [a * c, b * c] \\
I_{ab\bar{c}}(x) &\equiv x \in [a * (1-c), b * (1-c)]
\end{aligned}
$$

Stability Proof



Conclusions

- Interesting exercise; we learned a lot!
- The full language is basically done.
- We need your help! Let us know what would be interesting to check!
- Most complaints about plugins cannot be solved by verification.
- We are investigating a slightly different approach.
- Working on linear systems theory and frequency domain properties.

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